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Application of the Hard-Core Boson Formalism to the Heisenberg Ferromagnet*

J. F. Cooke and H. H. Hahn†
Solid State Division, Oak Ridge National Laboratory, Oak Ridge, Tennessee 37830
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The rigorous method developed in a previous paper for the calculation of the time-dependent properties of localized spin systems is applied to the case of the arbitrary-spin isotropic Heisenberg model of ferromagnetism. A low-density approximation is used to calculate the spin Green's function $G_1(1;1')=-i\langle TS^-(1)S^+(1')\rangle$ within the framework of this formalism. For the special case of $S=\frac{1}{2}$, it is explicitly demonstrated that the nonphysical states produce errors that may not be exponentially small, and that in the hard-core limit these terms disappear. The results of this work prove that for arbitrary S the "truncated" version of the Holstein-Primakoff transformation written in normal product form will produce the correct low-temperature results to all orders in $1/2S$. The form of the space-time transform $G_1(p;z)$ of $G_1(1;1')$ that we obtain is different from that suggested by previous work, since it does not contain any function of p or z in the numerator. In fact, if our result is expanded at temperatures low enough, the first term in the expansion is exactly the result obtained for $G_1(p;z)$ by Silberglitt and Harris from the Dyson transformation by neglecting the effects of the nonphysical states.

I. INTRODUCTION

In a previous paper,[1] which will be referred to as I, we outlined a method for calculating the thermodynamic properties of localized spin systems by using the known techniques of boson many-body theory. For instance, it was shown in I that the thermal properties of the spin-$\frac{1}{2}$ isotropic Heisenberg model could be determined rigorously from the thermal properties of a boson system with only two-body interactions.

This paper will be concerned with the application of the formalism outlined in I to the special case of the arbitrary-spin isotropic Heisenberg model, which is described by the Hamiltonian

$$H=g\mu h\sum_{\mathbf{j}} S_{\mathbf{j}}{}^z-\sum_{\mathbf{j},\rho} J_\rho\{S_{\mathbf{j}}{}^+S_{\mathbf{j}+\rho}{}^-+S_{\mathbf{j}}{}^zS_{\mathbf{j}+\rho}{}^z\}\,,\quad J_\rho\geq 0\,, \tag{1}$$

$$S_{\mathbf{j}}{}^\pm=S_{\mathbf{j}}{}^x\pm iS_{\mathbf{j}}{}^y\,,$$

where the first term represents the Zeeman energy due to the external field h, and the second term represents the exchange interaction between the spins. The J_ρ are the exchange constants, and the sum on ρ represents a sum over all the neighbors of $\mathbf{j}$.

There have been many attempts at predicting the thermodynamic properties of the Heisenberg ferromagnet from a boson formalism. The initially successful approaches were based on the calculation of the spin system partition function, from which the static properties of the spin system could be calculated. Later on, the concepts developed in these calculations were used in efforts to calculate the spin Green's functions, from which the time-dependent as well as the static properties of the spin system could be calculated.

The first really successful approach to calculating the thermodynamic properties of the Heisenberg ferromagnet from a boson formalism was due to Dyson.[2] His approach was to establish a rigorous identity between the spin system partition function and a quantity resembling the partition function for a boson system described by a non-Hermitian boson Hamiltonian. The difference between this quantity and the partition function for the boson system was due to the proper exclusion of the effects of the nonphysical boson states. Dyson then argued that the contribution of the nonphysical states to the boson partition function was exponentially small at low temperature, and that he could therefore calculate the spin partition function by simply calculating the boson partition function. Even though there was an internal inconsistency in his argument,[3] Dyson was able to calculate the magnetization and specific heat correct to order T^4.

The best that has been done to date with the Hermitian Holstein-Primakoff boson theory is essentially to reproduce Dyson's results only to order $1/2S$.[4,5] The approach is to use the Holstein-Primakoff[6] transformation and expand it out in powers of $1/2S$, which is essentially an infinite-series expansion in the number operator, and to retain only those terms in the calculation of the partition function that are first order in $1/2S$, while neglecting nonphysical state effects.

Subsequently, other attempts at calculating the spin partition function have been made. In particular, an approach due to Morita,[7] which was discovered after the completion of this work, appears to use a boson transformation similar to the one we have suggested to calculate the grand partition function for the boson system. Another approach, due to Greenberg,[8] calculates the thermodynamic properties by means of a binary-kernel technique. Both of these methods are able to cope successfully with the kinematic interaction and to reproduce Dyson's results. However, they are both time-independent formalisms.

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† Visiting scientist from Kernforschungsanlage Jülich, Germany.
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The nonphysical state problems encountered in the partition-function approach are also manifested in the Green's-function methods. The universal assumption is that nonphysical state effects can be neglected. Once this assumption is made, spin Green's functions can be calculated from corresponding boson Green's functions.

The most successful attempts at calculating spin Green's functions in this manner are based on the concept used by Dyson in the calculation of the partition function. Tahir-Kheli and Ter Haar[9] followed this approach and obtained agreement with Dyson's results only to order $1/2S$. The reason for this was due to the termination used to decouple their Green's-function equations. Recently, Silberglitt and Harris[10] have pointed out that one must use a termination that is at least equivalent to the T-matrix theory in order to reproduce Dyson's results. The termination used by Tahir-Kheli and Ter Haar[9] essentially represents the first two terms in the infinite series of terms introduced by the T-matrix formalism.

It is the purpose of this paper to investigate the assumption made in the above-mentioned work that nonphysical state effects can be neglected, and to indicate what changes a proper treatment of these effects would have on the results for the spin Green's functions. Since the work of Silberglitt and Harris is the only work that we are familiar with that can reproduce Dyson's exact results from a boson Green's-function formalism, we have chosen to compare our results, which explicitly eliminate the effects produced by the nonphysical states, with theirs, which neglect these effects.

We will show that our Hermitian boson formalism developed in I, in which we use a transformation that simply amounts to a truncated version of the Holstein-Primakoff tranfsormation expanded in normal product form rather than in terms of the number operator, will reproduce Dyson's results to all orders in $1/2S$. We will also show that the result that Silberglitt and Harris obtain for the space-time transform of the spin Green's function $G_1(1;1')$ given in (16) is exactly the first term in an expansion of our result at low temperatures and at points well away from the bound-state spin-wave energies.

In Secs. II and III, we develop the boson system and indicate how the spin Green's function $G_1(1;1')$, given in (16), can be determined from Green's functions calculated for this boson system. Section IV is concerned with setting up the calculation of $G_1(1;1')$ in terms of the T-matrix approximation, and in Sec. V we obtain our results for the Green's function and discuss some of the results that can be obtained from it. We will attempt to present only the basic mathematical steps, since most of the intermediate mathematical steps are straightforward and rather lengthy to write down.

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II. BOSON SYSTEM

In accordance with the prescription outlined in I, we will construct a boson system suitable for the calculation of the thermodynamic properties of the arbitrary-spin isotropic Heisenberg ferromagnet. The spin-image operators to be used in this connection are[1]

$$\begin{aligned}
\tilde{S}_{\mathbf{j}}^{+} &= (2S)^{1/2} \sum_{\nu=0}^{2S} A_\nu(S)(b_{\mathbf{j}}^\dagger)^{\nu+1} b_{\mathbf{j}}^{\nu}, \\
\tilde{S}_{\mathbf{j}}^{-} &= (2S)^{1/2} \sum_{\nu=0}^{2S} A_\nu(S)(b_{\mathbf{j}}^\dagger)^{\nu} b_{\mathbf{j}}^{\nu+1} = (\tilde{S}_{\mathbf{j}}^{+})^\dagger, \\
\tilde{S}_{\mathbf{j}}^{z} &= -S + b_{\mathbf{j}}^\dagger b_{\mathbf{j}}, \\
A_\nu(S) &= \sum_{\mu=0}^{\nu} \frac{(-1)^{\mu+\nu}}{\mu!(\nu-\mu)!}\left(1-\frac{\mu}{2S}\right)^{1/2},
\end{aligned} \tag{2}$$

where S is the spin of the system.

The image of the spin Hamiltonian given in (1) is

$$\begin{aligned}
\tilde{H} &= g\mu h \sum_{\mathbf{j}} \tilde{S}_{\mathbf{j}}^{z} - \sum_{\mathbf{j},\rho} J_\rho \{\tilde{S}_{\mathbf{j}+\rho}^{+}\tilde{S}_{\mathbf{j}}^{-} + \tilde{S}_{\mathbf{j}+\rho}^{z}\tilde{S}_{\mathbf{j}}^{z}\} \\
&= E_0 + \tilde{H}_0(S) + \tilde{H}_2(S) + \sum_{n=3}^{4S+1} \tilde{H}_n(S),
\end{aligned} \tag{3}$$

where E_0 is a constant, $\tilde{H}_n$ describes n-body interactions, and, in terms of the Fourier transforms of b and $b^\dagger$ defined by

$$b_{\mathbf{j}} = (1/\sqrt{N}) \textstyle\sum_{k} b_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{j}}, \tag{4}$$

$$\tilde{H}_0(S) = \textstyle\sum_{\mathbf{k}} E_{\mathbf{k}}^0 b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, \tag{5}$$

$$\tilde{H}_2(S) = (1/2N) \sum_{\mathbf{k},\mathbf{k}',\mathbf{K}} V_{\mathbf{K}}(\mathbf{k},\mathbf{k}') \times b_{\mathbf{K}/2+\mathbf{k}}^\dagger b_{\mathbf{K}/2-\mathbf{k}}^\dagger b_{\mathbf{K}/2+\mathbf{k}'} b_{\mathbf{K}/2-\mathbf{k}'}, \tag{6}$$

where

$$E_{\mathbf{k}}^0 = g\mu h + 2S[J(0) - J(\mathbf{k})] = \eta/2 - 2SJ(\mathbf{k}), \tag{7}$$

$$\eta = 2g\mu h + 4SJ(0), \tag{8}$$

$$V_{\mathbf{K}}(\mathbf{k},\mathbf{k}') = a(S)[J_{\mathbf{K}}(k) + J_{\mathbf{K}}(k')] - J(\mathbf{k}+\mathbf{k}') - J(\mathbf{k}-\mathbf{k}'), \tag{9}$$

$$J_{\mathbf{K}}(\mathbf{k}) = J(\tfrac{1}{2}\mathbf{K}+\mathbf{k}) + J(\tfrac{1}{2}\mathbf{K}-\mathbf{k}), \tag{10}$$

$$a(S) = 2SA_1(S) = 2S[1-\sqrt{(1-1/2S)}], \tag{11}$$

$$J(\mathbf{k}) = \textstyle\sum_{\rho} J_\rho e^{i\mathbf{k}\cdot\rho}. \tag{12}$$

In contrast to other methods, the image operator $\tilde{H}$ cannot be taken as the Hamiltonian for the boson system. Instead, we must construct the Hamiltonian $\hat{H}$ for the boson system, so that it is the simplest Hamiltonian that satisfies the conditions (I 40)–(I 42).[1]

For $S=\frac{1}{2}$, we have shown[1] that the simplest choice for $\hat{H}$ is

$$\hat{H} = E_0 + \tilde{H}_0(\tfrac{1}{2}) + \tilde{H}_2(\tfrac{1}{2}) + V(\tfrac{1}{2}), \tag{13}$$

where

$$V(\tfrac{1}{2})=\tfrac{1}{2}v_0\sum_{\mathbf{j}}(b_{\mathbf{j}}^\dagger)^2 b_{\mathbf{j}}^2$$
$$=(v_0/2N)\sum_{\mathbf{k},\mathbf{k}',\mathbf{K}} b_{\mathbf{K}/2+\mathbf{k}}^\dagger b_{\mathbf{K}/2-\mathbf{k}}^\dagger b_{\mathbf{K}/2+\mathbf{k}'} b_{\mathbf{K}/2-\mathbf{k}'}. \quad (14)$$

For $S>\frac{1}{2}$, the simplest $\hat{H}$ can always be written in the form

$$\hat{H}(S)=E_0+\tilde{H}_0(S)+\tilde{H}_2(S)+\Delta H, \quad (15)$$

where ΔH contains only those terms that describe n-body interactions with $n\geq 3$. Thus, in particular, ΔH contains the term $V(S)$, which we use to eliminate the effects of the nonphysical states.

III. SPIN GREEN'S FUNCTION

One of the most important Green's functions for this spin system is defined by

$$G_1(1;1')=-i\langle TS^-(1)S^+(1')\rangle, \quad 0\leq it,\ it'\leq\beta$$
$$1 \text{ represents } \mathbf{j}_1,t_1, \quad (16)$$

since it allows us to calculate such static properties as the specific heat and the magnetization, as well as the inelastic magnetic scattering cross section for the scattering of slow neutrons from the ferromagnet.

This Green's function can be calculated from the boson many-body Green's functions, which are calculated with respect to $\hat{H}$ by means of the identity[1]

$$G_1(1;1')=\lim_{v_0\to\infty}\hat{G}_1(1;1'), \quad (17)$$

$$\hat{G}_1(1;1')=-i\langle T\tilde{S}^-(1)\tilde{S}^+(1')\rangle, \quad 0\leq it,\ it'\leq\beta$$
$$=2S\Gamma_1(1;1')-ia(S)[\Gamma_2(1,1';1'^+1'^+)$$
$$+\Gamma_2(1,1;1^+1')]+\text{higher-order many-body Green's functions}, \quad (18)$$

where

$$\Gamma_n(1,2,\ldots,n;1',2',\ldots,n')$$
$$=(-i)^n\langle Tb(1)\cdots b(n)b^+(1')\cdots b^+(n')\rangle_{\hat{H}} \quad (19)$$

and where $a(S)$ is defined in (11).

For $S=\frac{1}{2}$, $\hat{H}$ describes a system of bosons with two-body interactions. In I it was shown that

$$G_1(1;1')=\lim_{v_0\to\infty}\Gamma_1(1;1'), \quad 0<it,\ it'<\beta, \quad (20)$$

and thus we need to calculate only the one-body boson Green's function.

In order to proceed beyond this point for $S>\frac{1}{2}$, we must restrict the calculations to low enough temperatures so that the number of bosons is so small that we can neglect the effects of the n-body interactions for $n\geq 3$. That is, we introduce a low-density two-body approximation. This approximation allows us to neglect the contributions from the term $V(S)$, which we use as a tool to remove the effects of the nonphysical states from the calculation. In other words, for $S>\frac{1}{2}$, we expect there to be a temperature range for which the contributions of the nonphysical states are negligible.

Since the two-body interaction $V_K(k,k')$ given in (9) is not weak, we must use an appropriate low-density approximation such as the T-matrix approximation. This approximation will enable us to calculate the one- and two-body Green's functions appearing in (18) in the temperature range in which only the two-body interaction is important.

In order to obtain the low-temperature results for $S=\frac{1}{2}$, we must also use the T-matrix approximation, since we must treat correctly the hard-core limit $v_0\to\infty$.[1]

IV. T-MATRIX APPROXIMATION

Since the T-matrix approximation has been discussed in detail by several authors,[11,12] we will only present the results that are necessary for the calculation of the one- and two-body Green's functions.

The results we need are most easily expressed in terms of the space-time Fourier transform of the functions involved. The pure imaginary-time transforms are defined by

$$F(t-t')=(1/-i\beta)\textstyle\sum_\nu e^{-iz_\nu(t-t')}F(z_\nu),$$
$$0\leq it,\ it'\leq\beta, \quad (21)$$
$$z_\nu=\pi\nu/-i\beta,$$

where ν is an even integer, and

$$\beta=1/k_BT, \quad (22)$$

where k_B is Boltzmann's constant and T is temperature. The space transforms with respect to the spatial variables are defined by

$$F(\mathbf{j})=(1/N)\textstyle\sum_{\mathbf{k}} F(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{j}}, \quad (23$$

$$F(\mathbf{j}_1,\mathbf{j}_2;\mathbf{j}_1',\mathbf{j}_2')=(1/N)\textstyle\sum_{\mathbf{K}} F_{\mathbf{K}}(\mathbf{j}_1-\mathbf{j}_2,\mathbf{j}_1'-\mathbf{j}_2')$$
$$\times\exp[-i\tfrac{1}{2}\mathbf{K}(\mathbf{j}_1+\mathbf{j}_2-\mathbf{j}_1'-\mathbf{j}_2')], \quad (24)$$

$$F_{\mathbf{K}}(\mathbf{j};\mathbf{j}')=(1/N^2)\sum_{\mathbf{k},\mathbf{k}'}F_{\mathbf{K}}(\mathbf{k},\mathbf{k}')e^{i\mathbf{k}\cdot\mathbf{j}}e^{-i\mathbf{k}'\cdot\mathbf{j}'}. \quad (25)$$

The space-time transform of $\Gamma_2(1,1';1'^+1'^+)$ will be denoted by $\Gamma_{2a}(\mathbf{p};z_\nu)$, and that of $\Gamma_2(1,1;1^+1')$ by $\Gamma_{2b}(\mathbf{p};z_\nu)$.

It is well known from Dyson's equation that the space-time transform of the one-body boson Green's function can be written in the form

$$\Gamma_1(\mathbf{p};z_\nu)=1/(z_\nu-E_{\mathbf{p}}^0-\Sigma(\mathbf{p};z_\nu)), \quad (26)$$

where $\Sigma(\mathbf{p};z_\nu)$ is the self-energy. The T-matrix approximation then gives us the following approximations for

[11] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1963).

[12] L. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1962).

Σ, Γ_{2a}, and Γ_{2b}:

$$\Sigma(\mathbf{p};z_\nu)=-(2/N\beta)\sum_{\nu',\mathbf{q}}\Gamma_1(\mathbf{q};z_{\nu'})\times T_{\mathbf{p}+\mathbf{q}}(\tfrac{1}{2}(\mathbf{q}-\mathbf{p}),\tfrac{1}{2}(\mathbf{q}-\mathbf{p});z_\nu+z_{\nu'}),\quad(27)$$

$$\Gamma_{2l}(\mathbf{p};z_\nu)=-iW_l(\mathbf{p};z_\nu)\Gamma_1(\mathbf{p};z_\nu),\quad l=a,b,\quad(28)$$

$$W_a(\mathbf{p};z_\nu)=2\langle b^\dagger b\rangle-(2/N^2\beta)\sum_{\nu'}\sum_{\mathbf{q},\mathbf{l}}\Gamma_1(\mathbf{q};z_{\nu'})\times T_{\mathbf{p}+\mathbf{q}}(\tfrac{1}{2}(\mathbf{q}-\mathbf{p}),\mathbf{l};z_\nu+z_{\nu'})\times g_{\mathbf{p}+\mathbf{q}}(\mathbf{l};z_\nu+z_{\nu'}),\quad(29)$$

$$W_b(\mathbf{p};z_\nu)=2\langle b^\dagger b\rangle-(2/N^2\beta)\sum_{\nu'}\sum_{\mathbf{q},\mathbf{l}}g_{\mathbf{p}+\mathbf{q}}(\mathbf{l};z_\nu+z_{\nu'})\times T_{\mathbf{p}+\mathbf{q}}(\mathbf{l},\tfrac{1}{2}(\mathbf{q}-\mathbf{p});z_\nu+z_{\nu'})\Gamma_1(\mathbf{q};z_{\nu'}),\quad(30)$$

where

$$\langle b^\dagger b\rangle=\lim_{\epsilon\to+0}(-1/N\beta)\sum_{\mathbf{q},\nu'}\Gamma_1(\mathbf{q};z_{\nu'})e^{\epsilon z_{\nu'}}\quad(31)$$

and $g_{\mathbf{K}}(\mathbf{k},\mathbf{k}';z)=N\delta_{k,k'}g_{\mathbf{K}}(\mathbf{k};z)$ is the transform of

$$g(\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_1',\mathbf{j}_2';t_1-t_1')=i\Gamma_1(\mathbf{j}_1-\mathbf{j}_1';t_1-t_1')\times\Gamma_1(\mathbf{j}_2-\mathbf{j}_2';t_1-t_1').\quad(32)$$

The expressions for W_a and W_b cannot be directly found in the references cited; however, they follow in a simple manner from the results that are given in these references.

The T matrix to be used in the above equations is the solution of

$$T_{\mathbf{K}}(\mathbf{k},\mathbf{k}';z)=V_{\mathbf{K}}(\mathbf{k},\mathbf{k}')+(1/N)\sum_{\mathbf{l}}T_{\mathbf{K}}(\mathbf{k},\mathbf{l};z)\times g_{\mathbf{K}}(\mathbf{l};z)V_{\mathbf{K}}(\mathbf{l},\mathbf{k}'),\quad(33)$$

where V is the two-body interaction

$$V_{\mathbf{K}}(\mathbf{k},\mathbf{k}')=v_0(S)+a(S)[J_{\mathbf{K}}(\mathbf{k})+J_{\mathbf{K}}(\mathbf{k}')]-J(\mathbf{k}+\mathbf{k}')-J(\mathbf{k}-\mathbf{k}'),\quad(34)$$

$$v_0(S)=v_0,\quad S=\tfrac{1}{2}$$
$$=0,\quad S>\tfrac{1}{2}.\quad(35)$$

The form of V for $S=\frac{1}{2}$ is obtained from (13), and that for $S>\frac{1}{2}$ from (15) or (6). We have introduced the function $v_0(S)$ so that the cases $S=\frac{1}{2}$ and $S>\frac{1}{2}$ can be treated together.

It is clear that Eq. (33) represents an extremely complicated equation for the T matrix, since g depends implicitly on T through Eqs. (32), (26), and (27). The procedure that we will use to solve (33) is first to calculate a zeroth approximation for g by replacing Γ_1 by $\bar{\Gamma}_1$, the one-body Green's function for the noninteracting system whose space-time transform is

$$\bar{\Gamma}_1(\mathbf{p};z_\nu)=1/(z_\nu-E_{\mathbf{p}}{}^0),\quad(36)$$

which, from (32), (21), (24), and (25), gives

$$g_{\mathbf{K}}(\mathbf{k};z_\nu)\simeq[1+n_{\mathbf{K}}(\mathbf{k})]/[z_\nu-E_{\mathbf{K}}(\mathbf{k})],\quad(37)$$

$$E_{\mathbf{K}}(\mathbf{k})=E_{\mathbf{K}/2+\mathbf{k}}{}^0+E_{\mathbf{K}/2-\mathbf{k}}{}^0,\quad(38)$$

$$n_{\mathbf{K}}(\mathbf{k})=n(E_{\mathbf{K}/2+\mathbf{k}}{}^0)+n(E_{\mathbf{K}/2-\mathbf{k}}{}^0),\quad(39)$$

$$n(\omega)=1/(e^{\beta\omega}-1).\quad(40)$$

The T-matrix equation can now be solved to obtain $\bar{T}$, the zeroth approximation for T. This result can then be used to calculate a first approximation for Γ_1 by means of (26) and (27). This cycle can then be repeated until self-consistency is obtained.

We will calculate only the first approximation for Γ_1, Γ_{2a}, and Γ_{2b}, since these will correctly give the leading temperature contributions to the thermodynamic properties of the ferromagnet. In order to be consistent within this approximation, we must also replace Γ_1 by $\bar{\Gamma}_1$ in the expression for Σ, Γ_{2a}, and Γ_{2b}. When this is done we can use the dispersion relation[12] for the T matrix to show that the leading temperature terms are given by

$$\Sigma(\mathbf{p};z_\nu)=(2/N)\sum_{\mathbf{q}}n_{\mathbf{q}}\times\bar{T}_{\mathbf{p}+\mathbf{q}}(\tfrac{1}{2}(\mathbf{q}-\mathbf{p}),\tfrac{1}{2}(\mathbf{q}-\mathbf{p});z_\nu+E_{\mathbf{q}}{}^0),\quad(41)$$

$$\Gamma_{2a}(\mathbf{p};z_\nu)=\bar{\Gamma}_1(\mathbf{p};z_\nu)\{2\Phi_0(T)+(2/N^2)\sum_{\mathbf{q},\mathbf{l}}n_{\mathbf{q}}\bar{T}_{\mathbf{p}+\mathbf{q}}(\tfrac{1}{2}(\mathbf{q}-\mathbf{p}),\mathbf{l};z_\nu+E_{\mathbf{q}}{}^0)\times\bar{g}_{\mathbf{p}+\mathbf{q}}(\mathbf{l};z_\nu+E_{\mathbf{q}}{}^0)\},\quad(42)$$

$$\Gamma_{2b}(\mathbf{p};z_\nu)=\bar{\Gamma}_1(\mathbf{p};z_\nu)\{2\Phi_0(T)+(2/N^2)\sum_{\mathbf{q},\mathbf{l}}\bar{g}_{\mathbf{p}+\mathbf{q}}(\mathbf{l};z_\nu+E_{\mathbf{q}}{}^0)\times\bar{T}_{\mathbf{p}+\mathbf{q}}(\mathbf{l},\tfrac{1}{2}(\mathbf{q}-\mathbf{p});z_\nu+E_{\mathbf{q}}{}^0)n_{\mathbf{q}}\},\quad(43)$$

$$n_{\mathbf{q}}=(e^{\beta E_{\mathbf{q}}{}^0}-1)^{-1},\quad(44)$$

$$\Phi_0(T)=(1/N)\sum_{\mathbf{q}}n_{\mathbf{q}},\quad(45)$$

and where $\bar{T}$ and $\bar{g}$ are the values of T and g at zero temperature. That is,

$$\bar{g}_{\mathbf{K}}(\mathbf{k};z_\nu)=1/[z_\nu-E_{\mathbf{K}}(\mathbf{k})]\quad(46)$$

and $\bar{T}$ is the solution of

$$\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{k}';z_\nu)=V_{\mathbf{K}}(\mathbf{k},\mathbf{k}')+\frac{1}{N}\sum_{\mathbf{l}}\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{l};z_\nu)\frac{V_{\mathbf{K}}(\mathbf{l},\mathbf{k}')}{z_\nu-E_{\mathbf{K}}(\mathbf{l})}.\quad(47)$$

As a simple example of how the spin Green's functions are calculated, and as a check on the approximations that we have used, we will now check out the identity

$$\lim_{v_0\to\infty}\Gamma_2(1,1';1'^+1'^+)=0;\quad S=\tfrac{1}{2};\quad 0<it,\ it'<\beta\quad(48)$$

for $S=\frac{1}{2}$, which is a rigorous result[1] used to establish

(20). First, notice that by summing (47) over $\mathbf{k}'$ for $S=\frac{1}{2}$,

$$\frac{1}{N}\sum_{\mathbf{k}'}\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{k}';z_\nu)$$
$$=v_0+J_{\mathbf{K}}(\mathbf{k})+\frac{1}{N}\sum_{\mathbf{l}}\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{l};z_\nu)\frac{v_0+J_{\mathbf{K}}(\mathbf{l})}{z_\nu-E_{\mathbf{K}}(\mathbf{l})}$$
$$=v_0+J_{\mathbf{K}}(\mathbf{k})+\frac{1}{N}\sum_{\mathbf{l}}\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{l};z_\nu)+(v_0+\eta-z_\nu)\frac{1}{N}\sum_{\mathbf{l}}\frac{\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{l};z_\nu)}{z_\nu-E_{\mathbf{K}}(\mathbf{l})},\quad(49)$$

since from (38) and (7)

$$v_0+J_{\mathbf{K}}(\mathbf{k})=v_0+\eta-z_\nu+z_\nu-E_{\mathbf{K}}(\mathbf{k}).\quad(50)$$

Then

$$\frac{1}{N}\sum_{\mathbf{l}}\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{l};z_\nu)\bar{g}_{\mathbf{K}}(\mathbf{l};z_\nu)\equiv\frac{1}{N}\sum_{\mathbf{l}}\frac{\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{l};z_\nu)}{z_\nu-E_{\mathbf{K}}(\mathbf{l})}=\frac{v_0+J_{\mathbf{K}}(\mathbf{k})}{z_\nu-\eta-v_0}.\quad(51)$$

Hence, from (42) and (51),

$$\Gamma_{2a}(\mathbf{p},z_\nu)=-i\left(2\Phi_0(T)+\frac{2}{N}\sum_{\mathbf{q}}n_{\mathbf{q}}\frac{v_0+J_{\mathbf{p}}+J_{\mathbf{q}}}{z_\nu+E_{\mathbf{q}}^0-\eta-v_0}\right)\times\frac{1}{z_\nu-E_{\mathbf{p}}^0}\quad(52)$$
$$=-\frac{2i}{N}\sum_{\mathbf{q}}\frac{n_{\mathbf{q}}}{z_\nu+E_{\mathbf{q}}^0-\eta-v_0}.\quad(53)$$

Now from (21) and (53), we have

$$\Gamma_{2a}(\mathbf{p};t-t')=-(2/N)\sum_{\mathbf{q}}n_{\mathbf{q}}e^{-i(v_0+\eta-E_{\mathbf{q}}^0)(t-t')}\times[1+n(v_0+\eta-E_{\mathbf{q}}^0)],\quad it>it'\quad(54)$$
$$=-(2/N)\sum_{\mathbf{q}}n_{\mathbf{q}}n(v_0+\eta-E_{\mathbf{q}}^0)e^{-i(v_0+\eta-E_{\mathbf{q}}^0)(t-t')},\quad it<it'.\quad(55)$$

Clearly, for $0<it,\ it'<\beta$

$$\lim_{v_0\to\infty}e^{-i(v_0+\eta-E_{\mathbf{q}}^0)(t-t')}[1+n(v_0+\eta-E_{\mathbf{q}}^0)]=0,\quad it>it'\quad(56)$$

$$\lim_{v_0\to\infty}e^{-i(v_0+\eta-E_{\mathbf{q}}^0)(t-t')}n(v_0+\eta-E_{\mathbf{q}}^0)=0,\quad it<it',\quad(57)$$

since η and $E_{\mathbf{q}}^0$ are bounded. Therefore, since the sum over $\mathbf{q}$ is a finite sum, we have

$$\lim_{v_0\to\infty}\Gamma_{2a}(\mathbf{p};t-t')=0,\quad 0<it,\ it'<\beta.\quad(58)$$

However, since $\Gamma_{2a}(\mathbf{p};t-t')$ is independent of $\mathbf{p}$ in this approximation, the space transform $\Gamma_2(1,1';1'^{+}1'^{+})$ can be obtained by simply multiplying (54) and (55) by $\delta_{\mathbf{j},\mathbf{j}'}$. Thus, we have established (48).

If we had neglected the effects of the nonphysical states, that is, if we had put $v_0=0$, it is clear from (55) that for $it'>it$, $\Gamma_{2a}(\mathbf{p};t-t')$ is a sum of terms, each of which is proportional to

$$\frac{1}{e^{\beta[\eta/2+J(\mathbf{k})]}-1}\simeq e^{-[g\mu h+J(0)+J(\mathbf{k})]/k_BT},\quad T\text{ small}.\quad(59)$$

Thus, even in the limit of zero field and for arbitrary $\mathbf{k}$, each of the terms in the sum is exponentially small. These terms are contributions from the nonphysical states, since, if we had included the v_0 term, these terms would have disappeared in the limit $v_0\to\infty$. Therefore, for $it'>it$, the contributions of the nonphysical states are exponentially small, a result similar to that found by Dyson in the calculation of the partition functions.

For $it>it'$, the situation is different. From (54) it is clear that for $v_0=0$ each term in sum for Γ_{2a} is proportional to

$$e^{-|\epsilon|[g\mu h+J(0)+J(k)]};\quad 0<|\epsilon|<\beta,\quad \epsilon=i(t-t'),\quad(60)$$

since T is small. Thus for ϵ small, the inclusion of the nonphysical states produces significant errors in the calculation of $\Gamma_2(1,1';1'^{+}1'^{+})$ for $it>it'$. We will indicate in Sec. V how this affects the calculation of $G_1(1;1')$.

V. CALCULATION OF $G_1(p;z_\nu)$

In order to calculate $G_1(\mathbf{p};z_\nu)$ we must first calculate the zero-temperature T matrix. This result is given in the Appendix by Eqs. (A4) and (A5) in terms of the function $\tilde{T}$, which we have called the Dyson T matrix. We will first treat the case $S=\frac{1}{2}$ in detail and then indicate how the calculation goes for $S\sim\frac{1}{2}$.

For $S=\frac{1}{2}$, if we substitute (A4) into (41), we find

$$\Sigma(\mathbf{p};z_\nu)=-(z_\nu-E_{\mathbf{p}}^0)\left[2\Phi_0(T)-(z_\nu-E_{\mathbf{p}}^0)\times\frac{2}{N}\sum_{\mathbf{q}}\frac{n_{\mathbf{q}}}{z_\nu+E_{\mathbf{q}}^0-\eta-v_0}-\Lambda(\mathbf{p};z_\nu)\right]+\Sigma_D(\mathbf{p};z_\nu),\quad(61)$$

where the terms Σ_D and Λ are defined below for arbitrary spin:

$$\Sigma_D(\mathbf{p};z_\nu)=(2/N)\sum_{\mathbf{q}}n_{\mathbf{p}}\times\tilde{T}_{\mathbf{p}+\mathbf{q}}(\tfrac{1}{2}(\mathbf{q}-\mathbf{p}),\tfrac{1}{2}(\mathbf{q}-\mathbf{p});z_\nu+E_{\mathbf{q}}^0),\quad(62)$$

$$\Lambda(\mathbf{p};z_\nu)=-\frac{2}{N^2}\sum_{\mathbf{q},\mathbf{l}}n_{\mathbf{q}}\frac{\tilde{T}_{\mathbf{p}+\mathbf{q}}(\mathbf{l},\frac{1}{2}(\mathbf{q}-\mathbf{p});z_\nu+E_{\mathbf{q}}^0)}{z_\nu+E_{\mathbf{q}}^0-E_{\mathbf{p}+\mathbf{q}}(\mathbf{l})}.\quad(63)$$

The term Σ_D is the Dyson self-energy. That is, it represents the self-energy due to the interaction between Dyson's ideal spin waves.

Because of the simple dependence of Σ on v_0 it is possible to carry out the inverse transform of $\Gamma_1(\mathbf{p};z_\nu)$ and, in a manner similar to that used in the calculation of $\Gamma_2(1,1';1'^+1'^+)$, to eliminate the effects produced by the nonphysical states by taking the limit $v_0\to\infty$.

A similar calculation can be carried out for $S>\frac{1}{2}$. However, since there is no v_0 dependence, we follow the procedure consistent with this low-temperature theory, which is to retain only the leading temperature term obtained for $G_1(\mathbf{p};z_\nu)$.

It turns out that this procedure will produce the correct result for $S=\frac{1}{2}$ with $v_0=0$. That is, if we put $v_0=0$ and calculate $G_1(1;1')$ by means of the general result given in I, namely,

$$G_1(1;1')=\lim_{v_0\to\infty}[\Gamma_1(1;1')-i\Gamma_2(1,1';1'^+1'^+)],\quad S=\tfrac{1}{2}$$
$$\simeq\Gamma_1(1;1')-i\Gamma_2(1,1';1'^+1'^+),\quad v_0=0,\qquad(64)$$

we obtain significant contributions from the nonphysical states in both Γ_1 and Γ_2. However, when we take the difference indicated in (64), a cancellation takes place, and we obtain the correct low-temperature result for $G_1(1;1')$, that is, the result obtained by the hard-core limit.

A simpler method for obtaining the correct result for $S=\frac{1}{2}$ with $v_0=0$, which we will use for $S>\frac{1}{2}$, is first to calculate the space-time transforms of the one- and two-body boson Green's functions that are needed to calculate $G_1(1;1')$. These are given in (64) for $S=\frac{1}{2}$ and by (18) for $S>\frac{1}{2}$. For $S>\frac{1}{2}$, we neglect all higher-order boson Green's functions ($n\geq3$), since we are restricting the calculation to low temperatures. We then use the space-time transforms of (64) and (18) to obtain an equation for $G_1(p;z_\nu)$ in terms of the appropriate transforms of Γ_1 and Γ_2, and we then take only the leading temperature term of the result. For $S=\frac{1}{2}$, this is exactly the result obtained by the hard-core limit and, therefore, for $S>\frac{1}{2}$, we may be inadvertently eliminating the effects produced by the nonphysical states by simply taking only the leading temperature term for $G_1(p;z_\nu)$. It should be made clear that we use this method only for $S>\frac{1}{2}$, since for $S=\frac{1}{2}$, we can take the hard-core limit.

The result, for arbitrary spin, is given by

$$G_1(\mathbf{p};z_\nu)=\frac{2S}{(z_\nu-E_{\mathbf{p}}{}^0)[1+(1/S)\Phi_0(T)-(1/2S)\Lambda(\mathbf{p};z_\nu)]-\Sigma_D(\mathbf{p};z_\nu)},\qquad(65)$$

which, not too surprisingly, is the same result for $S=\frac{1}{2}$ as can be obtained by calculating $\lim_{v_0\to0}\Gamma_1(\mathbf{p};z_\nu)$. The fact that (65) is independent of the complicated spin-dependent function $a(S)$, for $S>\frac{1}{2}$, is a simple consequence of the identity (A13), namely,

$$a^2(S)-4Sa(S)+2S=0.\qquad(66)$$

Since (65) is a low-temperature result, $\Phi_0(T)\ll1$, and we could equally well have written

$$G_1(\mathbf{p};z_\nu)\simeq\frac{2S-2\Phi_0(T)}{(z_\nu-E_{\mathbf{p}}{}^0)[1-(1/2S)\Lambda(\mathbf{p};z_\nu)]-\Sigma_D(\mathbf{p};z_\nu)}\simeq\frac{2\langle S^z\rangle^0}{z_\nu-E_{\mathbf{p}}{}^0-\bar\Sigma(\mathbf{p};z_\nu)},\qquad(67)$$

where

$$\bar\Sigma(\mathbf{p};z_\nu)=(1/2S)(z_\nu-E_{\mathbf{p}}{}^0)\Lambda(\mathbf{p};z_\nu)+\Sigma_D(\mathbf{p};z_\nu)\qquad(68)$$

and

$$2\langle S^z\rangle^0\simeq2S-2\Phi_0(T)=2S-C_{3/2}T^{3/2}-\cdots\qquad(69)$$

gives the well-known leading temperature correction for the magnetization.

As long as the magnitudes of $\Phi_0(T)$ and Λ are small, we can expand (65) and (67) to first order in these quantities, obtaining

$$G_1(\mathbf{p};z_\nu)\simeq[2\langle S^z\rangle^0+\Lambda(\mathbf{p};z_\nu)]/[z_\nu-E_{\mathbf{p}}{}^0-\Sigma_D(\mathbf{p};z_\nu)],\qquad(70)$$

which, from (62) and (63), is exactly the same result obtained by Silberglitt and Harris[10] by using the Dyson-Maléev transformation and neglecting the effects produced by the nonphysical states. The fact that Λ appears in the denominator of our result, and not in the numerator, as in (70), has some interesting consequences.

First of all, it is clear that if Λ has any resonances, such as those produced by bound-state spin waves,[10] we cannot expand as we have done in (70) near these resonances. In particular, our expression for $G_1(\mathbf{p};z_\nu)$ is zero at energies ω_B, corresponding to a bound state between two spin waves, a result that is different from the approximate form given in (70).[10] We do, however, obtain poles in the Green's function that are related to the bound-state energies, which are in exact agreement with those found by Silberglitt and Harris.

Secondly, the form of the Green's function given in (65) or (67) leads to several interesting observations regarding the interpretation of neutron scattering experiments. The inelastic magnetic scattering cross section for slow neutrons is governed by the spectral function $\rho(\mathbf{p};\omega)$ given by

$$\rho(\mathbf{p};\omega)=i\lim_{\epsilon\to0}[G_1(\mathbf{p};\omega+i\epsilon)-G_1(\mathbf{p};\omega-i\epsilon)]\qquad(71)$$
$$=2\bar\Sigma''(\mathbf{p};\omega)/\{[\omega-E_{\mathbf{p}}{}^0-\bar\Sigma'(\mathbf{p};\omega)]^2+[\bar\Sigma''(\mathbf{p};\omega)]^2\},\qquad(72)$$

where

$$\lim_{\epsilon\to 0}\bar{\Sigma}(\mathbf{p};\omega\pm i\epsilon)=\bar{\Sigma}'(\mathbf{p};\omega)\pm i\bar{\Sigma}''(\mathbf{p};\omega)\,, \tag{73}$$

and where we make similar definitions for Λ and Σ_D. The solutions of

$$\omega-E_{\mathbf{p}}^0-\bar{\Sigma}'(\mathbf{p};\omega)=0 \tag{74}$$

locate the peak in the scattered intensity and give a renormalized "spin-wave" energy. However, from (68) we see that (74) becomes

$$(\omega-E_{\mathbf{p}}^0)[1-\Lambda'(\mathbf{p};\omega)/2S]-\Sigma_D'(\mathbf{p};\omega)=0\,, \tag{75}$$

or for low T,

$$\omega-E_{\mathbf{p}}^0-\Sigma_D'(\mathbf{p};\omega)=0\,. \tag{76}$$

That is, the low-temperature "spin-wave" renormalized energies are indentical to those predicted by the Dyson formalism. Similarly, the damping or the width of the "spin-wave" peak is given precisely by the imaginary part of the Dyson self-energy Σ_D, since the "spin-wave" peak occurs at $\omega\simeq E_p^0$ for low temperatures. That is, from (68),

$$\bar{\Sigma}''(\mathbf{p};\omega\simeq E_p^0)\simeq\Sigma_D''(\mathbf{p};\omega\simeq E_{\mathbf{p}}^0)\,. \tag{77}$$

Thus, at low temperatures we can regard $\bar{\Sigma}(\mathbf{p};z_\nu)$, given in (68), as being composed of two parts. One is the Dyson self-energy, which results from what Dyson called a dynamical interaction between the ideal spin waves, and the other part results from what he called the kinematic interaction. At low temperatures, the position of the "spin-wave" peak and the half-width (or damping) at the peak are given entirely by the dynamical part of the interaction. However, it is clear from (72) and (68) that, as the temperature increases, the location of the "spin-wave" peak and the half-width of the peak will be dependent on both the dynamic and the kinematic interactions. It is at this point that the concept of Dyson's ideal spin waves begins to lose its meaning.

Another point to be noticed here is that the spectral function, (72), is never a true Lorentzian. However, it is clear that if we stay away from the bound-state energies for which Λ has a resonance,[10] we can write $\rho(\mathbf{p};\omega)$ for temperatures low enough near the spin-wave peak as a product of a Lorentzian and the function $2S-2\Phi_0(T)+\Lambda'(\mathbf{p};\omega)\simeq 2\langle S^z\rangle^0+\Lambda'(\mathbf{p};\omega)$.

Finally, if we are interested in calculating the low-temperature thermodynamic properties of the ferromagnet, we can neglect the effects of the bound states, expand $G_1(\mathbf{p};z_\nu)$ as shown in (70), and take only the first term. As pointed out before, this approximation of our Green's function is precisely the Green's function found by Silberglitt and Harris,[10] who showed that it reproduces Dyson's low-temperature thermodynamic results to all orders in $1/2S$.

VI. CONCLUSIONS

It has been shown that, starting with a boson system described by a Hermitian Hamiltonian, it is possible to derive the correct low-temperature static properties of the Heisenberg ferromagnet to all orders in $1/2S$. In addition, our result for $G_1(\mathbf{p};z_\nu)$ is different in form from that predicted by previous boson formalisms,[10] as well as spin formalisms,[13–15] in that it contains no function of p or z in the numerator of the expression. Instead, the form of the result given in (67) suggests that $G_1(\mathbf{p};z_\nu)$ can be written as a product of a function of temperature, namely, $2\langle S^z\rangle$, and a function of $\mathbf{p}$ and z that is identical to that of the space-time transform of a one-body boson Green's function. In other words, it appears that at low temperatures we may write the spin Green's function $G_1(1;1')$ as a product of $2\langle S^z\rangle$ and a function that satisfies a Dyson-like equation. This result is in agreement with the results of a functional derivative approach in the spin-operator formalism due to Mills,[16] which proves that this observation is true for all temperatures, except possibly where $2\langle S^z\rangle=0$. The form that we obtain is also consistent with the result obtained by Tahir-Kheli,[17] which is based on a fugacity expansion.

We also pointed out that the result for $G_1(\mathbf{p};z_\nu)$ obtained by Silberglitt and Harris from the Dyson-Maléev transformation represents the first term in an expansion of our result in powers of the kinematic terms $\Phi_0(T)$ and $\Lambda(\mathbf{p};z_\nu)$, and that our result is different from theirs only near the bound-state energies ω_B. We have thus verified the assumption that, for the Heisenberg ferromagnet at low temperatures, the effects of the nonphysical states on the calculation of $G_1(\mathbf{p};z_\nu)$ can be neglected as long as the Dyson-Maléev transformation is used and the energies corresponding to the ω_B's are avoided.

This formalism also proves that the Holstein-Primakoff transformation can produce the correct low-temperature thermodynamic results if it is expanded out in normal product form and not in powers of the number operator, as is the usual case. The proof of this statement rests on the fact that the set of operators that we have used [given in (2)], which were derived from conditions imposed by our formalism,[1] corresponds exactly to expanding the Holstein-Primakoff transformation in normal product form and then truncating this expression after the $(2S+1)$ term. Therefore, the Holstein-Primakoff transformation expanded in normal product form will produce the same low-temperature result for $G_1(\mathbf{p};z_\nu)$ as ours, since in this temperature region we needed only a knowledge of the first two terms in such an expansion.

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APPENDIX

In this Appendix we determine the zero-temperature T matrix $\bar{T}$, given by (47), or,

$$\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{k}';z_\nu)=V_{\mathbf{K}}(\mathbf{k},\mathbf{k}')+(1/N)\sum_{\mathbf{l}}\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{l};z_\nu)\times V_{\mathbf{K}}(\mathbf{l},\mathbf{k}')/[z_\nu-E_{\mathbf{K}}(\mathbf{l})], \quad \text{(A1)}$$

where $V_{\mathbf{K}}(\mathbf{k},\mathbf{k}')$ is given by Eq. (34).

In order to present the solution in a simple and usable form, we will express it in terms of the zero-temperature Dyson T matrix,[3] $\tilde{T}$, which satisfies

$$\tilde{T}_{\mathbf{K}}(\mathbf{k},\mathbf{k}';z_\nu)=\tilde{V}_{\mathbf{K}}(\mathbf{k},\mathbf{k}')+(1/N)\sum_{\mathbf{l}}\tilde{T}_{\mathbf{K}}(\mathbf{k},\mathbf{l};z_\nu)\times \tilde{V}_{\mathbf{K}}(\mathbf{l},\mathbf{k}')/[z_\nu-E_{\mathbf{K}}(\mathbf{l})], \quad \text{(A2)}$$

where

$$\tilde{V}_{\mathbf{K}}(\mathbf{k},\mathbf{k}')=J_{\mathbf{K}}(\mathbf{k})-J(\mathbf{k}+\mathbf{k}')-J(\mathbf{k}-\mathbf{k}') \quad \text{(A3)}$$

represents the interaction between Dyson's ideal spin waves. Equation (A2) can easily be solved for $\tilde{T}$.[10]

The solution of (A1) can then be written in the form

$$\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{k}';z_\nu)=[z_\nu-E_{\mathbf{K}}(\mathbf{k})]\times\left(\frac{v_0+J_{\mathbf{K}}(\mathbf{k}')}{z-\eta-v_0}-\frac{1}{N}\sum_{\mathbf{l}}\frac{\tilde{T}_{\mathbf{K}}(\mathbf{l},\mathbf{k}';z_\nu)}{z_\nu-E_{\mathbf{K}}(\mathbf{l})}\right)+\tilde{T}_{\mathbf{K}}(\mathbf{k},\mathbf{k}';z_\nu),\quad S=\tfrac{1}{2} \quad \text{(A4)}$$

$$\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{k}';z_\nu)=[(a(S)-1)/(z_\nu-\eta)][z_\nu-E_{\mathbf{K}}(\mathbf{k}')]\times(1/N)\sum_{\mathbf{l}}R_{\mathbf{K}}(\mathbf{k},\mathbf{l};z_\nu)+R_{\mathbf{K}}(\mathbf{k},\mathbf{k}';z_\nu),\quad S>\tfrac{1}{2}, \quad \text{(A5)}$$

where

$$R_{\mathbf{K}}(\mathbf{k},\mathbf{k}';z_\nu)=\frac{a(S)}{2S}[z_\nu-E_{\mathbf{K}}(\mathbf{k})]\times\left(\frac{2SJ_{\mathbf{K}}(\mathbf{k}')}{z_\nu-\eta}-\frac{1}{N}\sum_{\mathbf{l}}\frac{\tilde{T}_{\mathbf{K}}(\mathbf{l},\mathbf{k}';z_\nu)}{z_\nu-E_{\mathbf{K}}(\mathbf{l})}\right)+\tilde{T}_{\mathbf{K}}(\mathbf{k},\mathbf{k}';z_\nu). \quad \text{(A6)}$$

It can be verified by direct substitution that (A4) and (A5) are indeed the solutions of (A1) for $S=\frac{1}{2}$ and $S>\frac{1}{2}$, respectively. We will prove this for $S=\frac{1}{2}$. The proof for $S>\frac{1}{2}$ is similar, but longer to write out.

First, it is convenient to rewrite (A1) by using (51) and (A3). Then, for $S=\frac{1}{2}$,

$$\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{k}';z_\nu)=\tilde{V}_{\mathbf{K}}(\mathbf{k},\mathbf{k}')+[v_0+J_{\mathbf{K}}(\mathbf{k}')]\left(1+\frac{1}{N}\sum_{\mathbf{l}}\frac{\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{l};z_\nu)}{z_\nu-E_{\mathbf{K}}(\mathbf{l})}\right)+\frac{1}{N}\sum_{\mathbf{l}}\frac{\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{l};z_\nu)}{z_\nu-E_{\mathbf{K}}(\mathbf{l})}\tilde{V}_{\mathbf{K}}(\mathbf{l},\mathbf{k}') \quad \text{(A7)}$$

or

$$\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{k}';z_\nu)=\tilde{V}_{\mathbf{K}}(\mathbf{k},\mathbf{k}')+\frac{[v_0+J_{\mathbf{K}}(\mathbf{k}')][z_\nu-E_{\mathbf{K}}(\mathbf{k})]}{z_\nu-\eta-v_0}+\frac{1}{N}\sum_{\mathbf{l}}\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{l};z_\nu)\frac{\tilde{V}_{\mathbf{K}}(\mathbf{l},\mathbf{k}')}{z_\nu-E_{\mathbf{K}}(\mathbf{l})}. \quad \text{(A8)}$$

It can be proved by summing (A8) over $\mathbf{k}'$ that the solution of (A8) automatically satisfies (51) and therefore we need only solve (A8) to find $\bar{T}$.

Now from (A4),

$$\frac{1}{N}\sum_{\mathbf{l}}\bar{T}_{\mathbf{K}}(\mathbf{k},\mathbf{l};z_\nu)\frac{\tilde{V}_{\mathbf{K}}(\mathbf{l},\mathbf{k}')}{z_\nu-E_{\mathbf{K}}(\mathbf{l})}=\frac{1}{N}\sum_{\mathbf{l}}\frac{\tilde{T}_{\mathbf{K}}(\mathbf{k},\mathbf{l};z_\nu)}{z_\nu-E_{\mathbf{K}}(\mathbf{l})}\tilde{V}_{\mathbf{K}}(\mathbf{l},\mathbf{k}')-[z_\nu-E_{\mathbf{K}}(\mathbf{k})]\frac{1}{N}\sum_{\mathbf{l}}\frac{\tilde{T}_{\mathbf{K}}(\mathbf{l},\mathbf{k}';z_\nu)}{z_\nu-E_{\mathbf{K}}(\mathbf{l})}, \quad \text{(A9)}$$

since, from (A2),

$$\frac{1}{N^2}\sum_{\mathbf{l},\mathbf{l}'}\left(\frac{\tilde{T}_{\mathbf{K}}(\mathbf{l},\mathbf{l}';z_\nu)}{z_\nu-E_{\mathbf{K}}(\mathbf{l})}\right)\left(\frac{\tilde{V}_{\mathbf{K}}(\mathbf{l}',\mathbf{k}')}{z_\nu-E_{\mathbf{K}}(\mathbf{l}')}\right)=\frac{1}{N}\sum_{\mathbf{l}}\frac{1}{z_\nu-E_{\mathbf{K}}(\mathbf{l})}\times[\tilde{T}_{\mathbf{K}}(\mathbf{l},\mathbf{k}';z_\nu)-\tilde{V}_{\mathbf{K}}(\mathbf{l},\mathbf{k}')], \quad \text{(A10)}$$

and, from (A3),

$$\frac{1}{N}\sum_{\mathbf{l}}\tilde{V}_{\mathbf{K}}(\mathbf{l},\mathbf{k}')=0. \quad \text{(A11)}$$

Then the right-hand side of (A8) becomes

$$\tilde{V}_{\mathbf{K}}(\mathbf{k},\mathbf{k}')+\frac{1}{N}\sum_{\mathbf{l}}\frac{\tilde{T}_{\mathbf{K}}(\mathbf{k},\mathbf{l};z_\nu)}{z_\nu-E_{\mathbf{K}}(\mathbf{l})}\tilde{V}_{\mathbf{K}}(\mathbf{l},\mathbf{k}')+[z_\nu-E_{\mathbf{K}}(\mathbf{k})]\times\left(\frac{v_0+J_{\mathbf{K}}(\mathbf{k}')}{z_\nu-\eta-v_0}-\frac{1}{N}\sum_{\mathbf{l}}\frac{\tilde{T}_{\mathbf{K}}(\mathbf{l},\mathbf{k}';z_\nu)}{z_\nu-E_{\mathbf{K}}(\mathbf{l})}\right), \quad \text{(A12)}$$

which, from (A2), is exactly the solution given in (A4).

It should be noted that the terms involving v_0 separate out in the zero-temperature spin-$\frac{1}{2}$ T matrix in terms of a simple function, which makes the treatment of the v_0 terms trivial.

The proof for $S>\frac{1}{2}$ proceeds in a similar way. It is necessary, however, to use the identity

$$a^2(S)-4Sa(S)+2S=0, \quad \text{(A13)}$$

which can easily be derived from the definition, Eq. (11),

$$a(S)=2S[1-(1-1/2S)^{1/2}]. \quad \text{(A14)}$$